CHAPTER 4: NET PRESENT VALUE

(Assigned problems are 1, 2, 7, 8, 11, 16, 23, 25, 28, 29, 31, 33, 36, 41, 42, 46, 50, and 52)

The title of this chapter may be "Net Present Value", but it is essentially an introduction (or review if you have had a prior finance course) of the Time Value of Money concepts. Nevertheless, the material covered in this chapter is important and essential in order to understand the content of this course. Students entering this course have a high variance of financial knowledge. Some have taken finance courses and others are taking their first finance course. As a result, these notes are written to accommodate all.

One period models (for now, *periods* are one year units)**:**

Example 1: You have \$1000 cash now (t=0) $\rightarrow$ therefore, the Present Value or PV_0 = \$1000. Let the cost of capital or interest rate be r=11% per year. What is the Future Value or FV_1 of this amount exactly one year from today at t=1? Here we say that n=1 year.

$FV_1 = PV_0[1 + r] = 1000[1 + 0.11] =$ **\$1110.00**

Practical interpretation: \$1000 in a bank account earning r=11% will earn \$110 interest over one year. Final amount in account is \$1110.00.

Example 2: You want to have exactly \$1000 in your account exactly one year from today. The account earns an interest rate r=9% per year. How much must you have in this account today (t=0) in order to have \$1000 at t=1.

We thus have FV_1 = \$1000. Given r=9% and n=1 year, we must fine PV_0.

From above: $FV_1 = PV_0[1 + r]$, rearrange to obtain $\rightarrow$ $PV_0 = FV_1/[1 + r]$

$PV_0 = FV_1/[1 + r] = 1000/[1 + 0.09] =$ **\$917.43** (this is known as *discounting*)

Practical interpretation: If \$917.43 earns r=9% for one year, it grows to \$1000.

Multiperiod examples (number of annual periods *n* is greater than 1)**:**

We have these following basic formulas:

$FV_n = PV_0[1 + r]^n$ (*compounding*)

$PV_0 = FV_n/[1 + r]^n$ (*discounting*)

$r = [FV_n/PV_0]^{1/n} - 1$

$n = \ln(FV_n/PV_0)/\ln(1+r)$

Note that all formulas on this page (single and multiperiod) pertain to *lump sum* amounts, i.e., one amount today at t=0 and one amount at a future time t=*n*, where n is measured in years, and *r* is an annual *effective* rate of interest.

Example 3: You deposit $500 into an account today that earns r=11% annual interest. The money is left in the account for n=5 years. What amount FV_5 is in the account 5 years from today?

$FV_n = PV_0[1 + r]^n \rightarrow FV_5 = 500[1 + 0.11]^5 = 500[1.6851] =$ **$842.53**

Note: $FV_5 = PV_0[1 + r][1 + r][1 + r][1 + r][1 + r] = PV_0[1 + r]^5$

Example 4: You own a bond that pays one future cash flow of $100,000 exactly 20 years from today. The applicable annual interest rate over this horizon is r=6.75%. What is this bond or financial claim worth today (t=0)?

Here, FV_{20} = $100,000 and n = 20 years. Find PV_0 by *discounting* the FV_{20} back to today (t=0).

$\rightarrow PV_0 = FV_n/[1 + r]^n = FV_{20}/[1 + 0.0675]^{20} = 100{,}000/3.69281 =$ **$27,079.61**

Note: $27,079.61 deposited into account for 20 years at r=6.75% will grow to $100,000 at t=20 years.

Example 5: A stock index is at 7900 today. It was at 800 exactly 15 years ago. What was the *annualized* rate of return for this 15-year horizon?

Here, FV_{15} = 7900, PV_0 = 800, and, of course, n = 15 years. Find *r*

$\rightarrow r = [FV_{15}/PV_0]^{1/15} - 1 = [7900/800]^{1/15} - 1 = 0.164937$ or **16.4937%**

Note: annualized return and geometric average (Chapter 9) are the same and are used to report investment performance (mutual funds report this in their prospectus). Also note that the Dow Jones Industrial Average or DJIA was 800 and 7900 in August 1982 and August 1997, respectively. Therefore, the annualized rate of return was 16.493% for this 15-year period. You can say that we are dealing with PV_{1982} and FV_{1997} in this example.

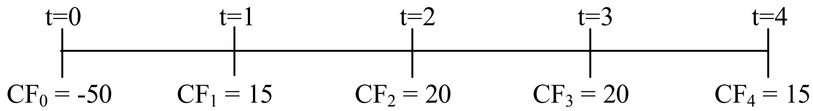
Calculator method of solving Example 5: Enter the following;

FV = 7900
PV = –800, the PV must be of the *opposite* sign of the future value, otherwise you will receive an error message. Think of the cash flows as having opposite flows; you *pay* 800 and later *receive* 7900. (it really does not matter which one is positive or negative here, just as long as you enter them into the calculator as opposite signs.
N = 15
P/Y = 1 interest compounding period per year (the calculator default is P/Y = 12)[1].
Solve for **I** = 16.493%

[1] Once you set P/Y, it stays there until you later manually change the setting or remove the batteries.

Streams of uneven cash flows:

The following time line illustrates five individual cash flows, beginning now at t=0 and ending at t=4 years. Let the applicable interest rate r=10% per year.



Example 6: Calculate the current **Present Value** or PV_0 of this stream of cash flows. Especially note that this cash flow stream resembles a business *project* that costs $50 now and produces four future cash flows.

We calculate the PV_0 of each of the individual five cash flows and then sum them to obtain the PV_0 of this uneven cash flow stream.

$$PV_0 = CF_0/(1+r)^0 + CF_1/(1+r)^1 + CF_2/(1+r)^2 + CF_3/(1+r)^3 + CF_4/(1+r)^4$$

$$PV_0 = -50/(1+0.1)^0 + 15/(1+0.1)^1 + 20/(1+0.1)^2 + 20/(1+0.1)^3 + 15/(1+0.1)^4$$

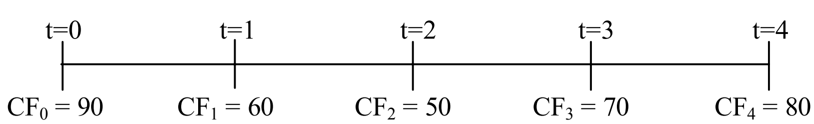
PV_0 = -50 + 13.6363 + 16.5289 + 15.0263 + 10.2452 =**$5.4368**

If this were a business project or real investment, then the Net Present Value or NPV would be $5.44. The interpretation is that the future cash *inflows* are $5.44 more valuable than the $50 cash *outflow* required today at t=0, i.e., you spend $50 today to create an asset that is instantly worth $55.44.

Calculator method: This is a major short cut. The procedure varies across calculators but resembles the following:

C0 = -50
C1 = 15
C2 = 20
C3 = 20
C4 = 15
Then enter I = 10%. Use the **NPV** key or function to obtain **$5.4368**

Example 7: Calculate the **Future Value** or FV_4 of the following stream of cash flows. Let the applicable interest rate r=7% per year.



One practical application of this example is to make five deposits into an investment account. The first deposit is made today (t=0) and the fifth and final deposit is made four years from today at t=4.

We just treat this problem as five separate investments into an account. In other words, we calculate the FV_4 of each of these five cash flows. The first deposit at t=0 earns annually compounded interest for n = 4 years, the second deposit at t=1 earns annually compounded interest for n = 3 years, and so on. The fifth and final deposit is made at t=4 and obviously earns zero interest.

$$FV_4 = CF_0(1+r)^4 + CF_1(1+r)^3 + CF_2(1+r)^2 + CF_3(1+r)^1 + CF_4(1+r)^0$$

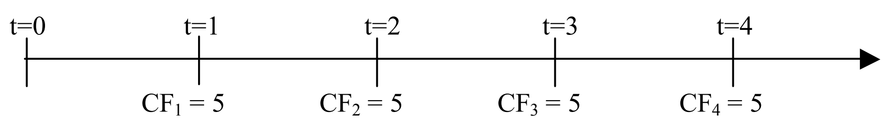
$$FV_4 = 90(1+0.07)^4 + 60(1+0.07)^3 + 50(1+0.07)^2 + 70(1+0.07)^1 + 80(1+0.07)^0$$

FV_4 = 117.9716 + 73.5026 + 57.2450 + 74.9000 + 80 = **$403.6192**

Calculator method: First find the PV_0 of the stream of cash flows. Next, take this PV_0 and compound it up to t=4 at r=7%: $FV_4 = PV_0[1+r]^4$. Calculators do not have any function that computes the FV of a cash flow stream.

Perpetuities:

A perpetuity is an *infinite* stream of constant cash flows, occurring at equal intervals. The following time line illustrates a perpetuity of $5 per year.



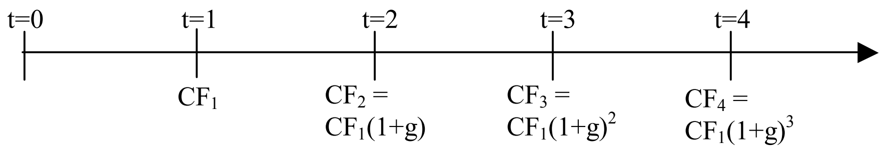
The formula for the Present Value of a perpetuity is: $\mathbf{PV_0 = CF/r}$. This equation calculates the PV of an infinite stream of equal cash flows.

Example 8: Calculate the current Present Value or PV_0 of the above perpetuity. Let the applicable interest rate r=6% per year.

$PV_0 = CF/r = 5/0.06 =$ **$83.3333**. Therefore, a share of *preferred stock* that pays a dividend of $5 per year should sell for $83.33 if the required rate of return is r=6% per year.

Growing perpetuity or Gordon Constant Growth Model:

This refers to a cash flow stream that grows at a constant annual growth rate *g*. The following time line illustrates such a cash flow stream.



Here, $CF_3 = CF_1(1+g)(1+g) = CF_1(1+g)^2$

The constant growth model formula is: $\mathbf{PV_0 = CF_1/(r-g)}$

Note the <u>timing difference</u> between the cash flow and the PV in the model! The cash flow CF_1 is *always* one period ahead of the PV_0 you are calculating.

The rate of return *r* must be greater than the annual cash flow growth rate *g* in order to use this formula. The growth rate *g* can be negative, as well.

The constant growth model is an extremely important formula and will be used extensively in this course. This time value of money formula or tool is used in nearly every stock valuation example. Note that if there is a cash flow CF_0 that has *not yet* been paid out, then use: $PV_0 = CF_0 + CF_1/(r-g)$

<u>Example 9</u>: Trillium is a mature corporation.[2] Trillium common stock has just paid a cash flow of CF_0 = \$2.00 per share.[3] An analyst estimates that Trillium's cash flows will grow at a constant annual growth rate of g=5.5% per year forever. Based on the risk of this stock, investors expect or require an annual rate of return of r=10% per year. Based on this estimate, what is this stock's current value?

<u>We must use</u>: $PV_0 = CF_1/(r-g)$. Note that CF_0 has been paid out and is *no longer* part of Trillium or it's stock price. The stock's value should be the Present Value of all *future expected* cash flows. CF_0 is *no longer* a future cash flow.

We need CF_1, but we are only given CF_0, r, and g. However, CF_1 is easily found:

$CF_1 = CF_0(1+g) = 2(1+0.055) =$ \$2.11 per share. Now we can find PV_0.

$PV_0 = CF_1/(r-g) = 2.11/(0.10 – 0.055) = 2.11/0.045 =$ **<u>\$46.89</u>** per share

[2] *Mature* firms are the only firms that can be considered to have a constant growth rate from today forward. However, when we cover stock valuation in Chapter 5, we state that *all* firms must mature someday.

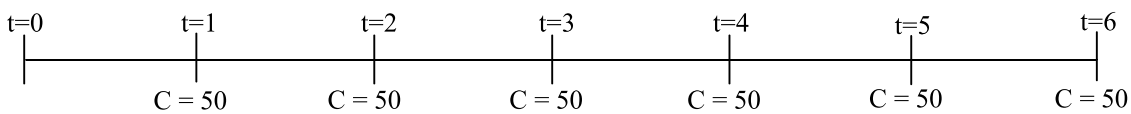
[3] Firms pay out cash in the form of (1) dividends and (2) stock repurchases. For stock valuation, what matters is how much can be paid out. How it is paid should not matter, as far as determining what the firm is worth.

An additional note concerning the constant growth model before we move on. Say that a firm will pay CF_{2020} in the year 2020. What will the firm be worth in 2019, just after the year 2019 cash flow is paid out?

Answer: it will be worth $PV_{2019} = CF_{2020}/(r-g)$. The one year timing difference between the PV and CF *never change* with this model!

Annuities:

An annuity is a finite stream of equal cash flows, occurring at equal intervals. It does not matter if the payments are being paid out or received, just as long as the payments are all in the same direction. The following is an example of a six-year (n=6) *ordinary* annuity consisting of six $50 payments.



Since this is a cash flow stream, the PV_0 and FV_6 can easily be found by using the procedure covered earlier for uneven cash flows. This can be long and tedious. However, since these annuity payments are *not* uneven, we have shortcuts for PV_0 and FV_n. The following two formulas are for the FV_n and PV_0 of an *ordinary* annuity. Note: with an *ordinary* annuity, the first cash flow is *always* at t=1.

$$PV_0 = C\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right]$$

$$FV_n = C\left[\frac{(1+r)^n}{r} - \frac{1}{r}\right]$$

Example 10: Let r=10% per year. Calculate the Present Value or PV_0 of the above six year annuity. Essentially, how much would you pay today (t=0) in order to receive this six year annuity of cash flows?[4]

$$PV_0 = C\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] = 50\left[\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^6}\right] = \underline{\mathbf{\$217.76}}$$

Long method: $PV_0 = 50/(1+0.1) + 50/(1+0.1)^2 + 50/(1+0.1)^3 + 50/(1+0.1)^4 + 50/(1+0.1)^5 + 50/(1+0.1)^6 = \$217.76$

Calculator method:

PMT = 50

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[4] You can also think of the PV_0 as how much you should deposit today in order to withdraw $50 for each of the next six years, so that the account is empty after the sixth or last $50 withdrawal.

I = 10%
N = 6
P/Y = 1
Solve for **PV** = -217.76 (PV will be the opposite sign of the payment)

Example 11: Let r=10% per year. Calculate the Future Value or FV_6 of the above six-year annuity. Essentially, if you make six $50 deposits into an account, the first and sixth (last) payments being made at t=1 and t=6 years, respectively, how much money will be in the account in six years (t=6)?

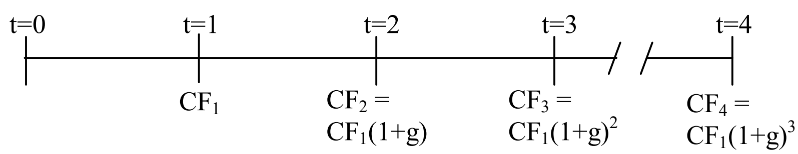
$$FV_n = C\left[\frac{(1+r)^n}{r} - \frac{1}{r}\right] = 50\left[\frac{(1+0.1)^6}{0.1} - \frac{1}{0.1}\right] = \underline{\mathbf{\$385.78}}$$

Calculator method:

PMT = -50 (sign does not really matter here, but FV will be of opposite sign)
I = 10%
N = 6
P/Y = 1
Solve for **FV** = 385.78 (positive here since PMT was made negative)

Growing annuities:

Basically, this is a *finite* series of cash flows that grows at a constant annual rate. While the Gordon constant growth model discussed earlier is infinite, the growing annuity is finite.



This concept is great for investment planning. The PV_0 and FV_n equations are:

$$PV_0 = C_1\left[\frac{1}{r-g}\right]\left[1-\left(\frac{1+g}{1+r}\right)^n\right]$$, where C_1 is the CF_1 on the above timeline

$$FV_n = C_1\left[\frac{1}{r-g}\right]\left[1-\left(\frac{1+g}{1+r}\right)^n\right](1+r)^n$$

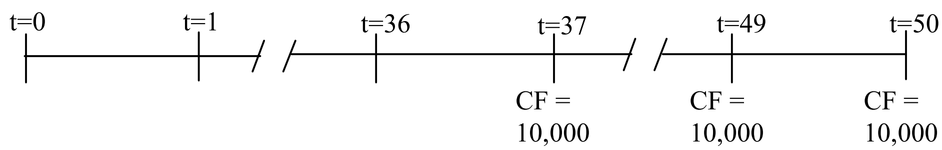
Example 12: Let r=11% per year. You deposit $2000 into an account exactly one year from today at t=1. The amount you deposit will grow by g=8% each year. You will make 30 yearly deposits, the final deposit is 30 years from today at t=30. How much money will be in the account at t=30 years?

$$FV_{30} = 2000\left[\frac{1}{0.11-0.08}\right]\left[1-\left(\frac{1+0.08}{1+0.11}\right)^{30}\right](1+0.11)^{30} = \underline{\mathbf{\$855{,}309.31}}$$

While this seems like a large amount, remember that this is an amount 30 years into the future, and it does not have the purchasing power of $855,309 today.

<u>Deferred annuities:</u> (highly prone to errors and mistakes!)

When the first payment of an annuity begins more than one period from now, we call it a deferred annuity. Let us consider the annuity on the following time line. We have an annuity of n=14 payments of $10,000 each. The first payment is 37 years from today, and the 14th and final payment is 50 years from today.



Calculate the current Present Value or PV_0 of this deferred annuity. There are actually several methods or procedures that will work here. I present the most common method below.

<u>Step 1</u>: Jump ahead to t=36 or 36 years from now. From the perspective of t=36 years, we would be looking at just an *n=14* year ordinary annuity with the first payment occurring one year or period ahead of that point in time. Calculate the Present Value of this 14 year annuity at t=36 years. Let r=8% per year.

$$PV_{36} = C\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] = 10{,}000\left[\frac{1}{0.08} - \frac{1}{0.08(1+0.08)^{14}}\right] = \underline{\mathbf{\$82{,}442.37}}$$

A good analogy is the following: Thirty six years from now (at t=36), you will need to have $82,442.37 in an account if you want to make 14 future withdrawals of $10,000 each, the first amount removed in one year and the last (14th) amount removed in 14 years (t=50).

<u>Step 2</u>: The Present Value PV_{36} from Step 1 becomes FV_{36} in this second step. This lump sum $82,442.37 amount from Step 1 must be discounted back exactly 36 years at r=8% per year in order to calculate PV_0.

$\rightarrow$ $PV_0 = FV_{36}/[1 + r]^{36} = 82{,}442.37/[1 + 0.08]^{36} =$ **<u>$5162.92</u>**

Other than annual interest compounding periods:

Prior to this point in these notes, interest rates were quoted as *effective* annual rates, and compounding periods were annual, e.g., r=9% per year.

However, home and car loans calculate the interest monthly. Credit cards calculate the interest daily. Many contracts use interest rates that *not* calculated annually.

What if you deposit $1000 today into an account that pays r=4% interest every six months or *semiannually*? After six months you will have $1000[1+0.04] = **$1040** in the account. After one year (two semiannual periods) you will have $1000[1+0.04][1+0.04] = **$1081.60** in the account.

It is uncommon to see *semiannual* interest rates quoted. However, we do know that this account pays 4% interest every six months, and the amount of money in this account does *effectively* grow by r=4% every six months.

The convention is to quote rates as *annual*, even though the compounding periods may not be annual. Here in our example, the r=4% *effective* semiannual rate is thus multiplied by the m=2 semiannual periods per year, and the interest rate will be quoted as 8% annual *nominal*, compounded *semiannually*.

For this particular example we state: (1) $\mathbf{r_{nom}=8\%}$ and (2) **m=2**. "r_{nom}" is what we call the nominal or *stated* interest rate, and "m" is the number of interest compounding periods per year. **Caveat**: while the 8% annual *nominal* rate is typically quoted, it is *not* the effective annual interest rate.

Note that: $r_{nom}/m = 8\%/2 = \mathbf{4\%}$. This 4% is the *effective semiannual* interest rate. The *effective annual* interest rate is (1+0.04)(1+0.04) – 1 = 0.0816 or **8.16%**, since here $1 grows to $1.0816 in one year.

Note that the following two equations relate the Present and Future lump sum values, where *m* is the number of compounding periods per year, and *n* is the number of years.

$$FV_n = PV_0\left[1+\frac{r_{nom}}{m}\right]^{n\cdot m} \quad \text{and} \quad PV_0 = \frac{FV_n}{\left[1+\frac{r_{nom}}{m}\right]^{n\cdot m}}$$

Example 13: You deposit $10,000 today into an account that pays 6% annual nominal interest, compounded quarterly. How much money will be in the account exactly 10 years from today if no further deposits are made?

The account will earn r_{nom}/m = 6%/4 = 1.5% interest every 3 months for $n \cdot m$ = (10)(4) = 40 quarterly periods.

$$FV_n = PV_0\left[1+\frac{r_{nom}}{m}\right]^{n \cdot m} = 10{,}000\left[1+\frac{0.06}{4}\right]^{10 \cdot 4} = \underline{\mathbf{\$18{,}140.18}}$$

Example 14: You want to have $20,000 in an account 5 years from today. The account pays 6% nominal interest, compounded quarterly. How much should you deposit today in order to have $20,000 in 5 years (n=20 quarterly payments)?

$$PV_0 = \frac{FV_n}{\left[1+\frac{r_{nom}}{m}\right]^{n \cdot m}} = \frac{20{,}000}{\left[1+\frac{0.06}{4}\right]^{5 \cdot 4}} = \underline{\mathbf{\$14{,}849.41}}$$

Converting nominal to effective rates:

In Examples 13 and 14, the interest rate was r_{nom} of 6% annual nominal, compounded quarterly. The effective quarterly interest rate is just r_{nom}/m = 6%/4 = 1.5% per quarter. The formula to convert nominal to effective rates is given as:

$$\text{Effective annual rate (EAR)} = \left[1+\frac{r_{nom}}{m}\right]^{m} - 1$$

In the case of Examples 13 and 14, calculate the actual effective annual rate of interest:

$$\text{Effective annual rate} = \left[1+\frac{r_{nom}}{m}\right]^{m} - 1 = \left[1+\frac{0.06}{4}\right]^{4} - 1 = 0.061364 \text{ or } \underline{\mathbf{6.1364\%}}$$

Also, to convert the annual *effective* rate to an annual *nominal* rate, given *m*:

$$r_{nom} = m\left[(r_{eff}+1)^{1/m} - 1\right]$$

Example 15: A credit card states that the annual interest rate charged is 18% APR (*annual percentage rate*) and that interest charges are calculated daily. APR is a nominal rate. In this example, the interest rate being quoted is 18% annual nominal, compounded daily. What is the annual effective rate of interest?

$$\text{Effective annual rate} = \left[1+\frac{r_{nom}}{m}\right]^{m} - 1 = \left[1+\frac{0.18}{365}\right]^{365} - 1 = 0.197164 \text{ or } \underline{\mathbf{19.7164\%}}$$

Example 16: $20,000 is borrowed today to finance a car purchase. The car loan is for 48 months. The loan contract states the interest rate as 10% APR, and the loan's interest charges and payments are made monthly. The amount of each monthly payment is identical. Calculate the effective annual interest rate and the monthly payment on this loan.

The interest rate quoted here is 10% annual nominal, compounded monthly.

$$\text{Effective annual rate} = \left[1+\frac{r_{nom}}{m}\right]^{m} - 1 = \left[1+\frac{0.10}{12}\right]^{12} - 1 = 0.104713 \text{ or } \underline{\mathbf{10.4713\%}}$$

Note: here the *effective monthly* interest rate is $r_{nom}/m = 10\%/12 = 0.83333\%$ per month.

The 48 month car loan is a 48 period annuity, consisting of 48 monthly payments or periods. Note the following annuity formula (it is similar to what we covered earlier in the introduction to annuities).

$$PV_0 = C\left[\frac{1}{r_{nom}/m} - \frac{1}{r_{nom}/m\left(1+r_{nom}/m\right)^{n\cdot m}}\right]$$

The amount borrowed today, \$20,000, is the PV_0.

$$20{,}000 = C\left[\frac{1}{0.1/12} - \frac{1}{0.1/12\left(1+0.1/12\right)^{48}}\right] \rightarrow 20{,}000 = C[39.42819] \rightarrow C = \underline{\mathbf{\$507.25}}$$

<u>Calculator method:</u>
PV = 20,000
I = 10% (the *nominal* annual rate of interest)
N = 48 (there are 48 periods or payments in this annuity's life)
P/Y = 12 (the calculator takes the i_{nom}=10% and divides it by *m* or P/Y=12 so it can solve the annuity using a monthly effective rate of 0.83333% per month)
Solve for **PMT** = -507.25 (negative here since PV was made positive)

<u>Alternative calculator method:</u>
PV = 20,000
I = 0.83333% (the *effective monthly* rate of interest)
N = 48 (there are 48 periods or payments in this annuity's life)
P/Y = 1
Solve for **PMT** = -507.25